J. Math. Anal. Appl. 410 (2014) 659–669

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Payne type inequalities for $L^p$-norms of the warping functions ☆

R.G. Salakhudinov

*Department of Mathematical Analysis, N.I. Lobachevsky Institute of Mathematics and Mechanics, Kazan Federal (Volga Region) University, Prof. Nuzhin str. 1/37, Kazan 420008, Russia*

ARTICLE INFO

*Article history:*
Received 28 May 2013
Available online 5 September 2013
Submitted by M. del Pino

*Keywords:*
Warping function
Torsional rigidity
Isoperimetric inequality
Payne's inequality
Isoperimetric monotonicity
Symmetrization

ABSTRACT

Let $u(x,G)$ be a warping function of a multiply connected plane domain $G$. A new physical functional of $u(x,G)$ with an isoperimetric monotonicity property is constructed. It is proved that $L^p$- and $L^q$-norms of the warping function satisfy sharp isoperimetric inequalities, which, besides the norms, can contain the functional $\mathbf{u}(G)=\sup_{x\in G}u(x,G)$. As a particular case of one of these inequalities it follows the classical result of Payne for the torsional rigidity of $G$. Our proofs are based on the technique of estimates on level lines devised by L.E. Payne.



## 1. Introduction

Let $G$ be a multiply connected plane domain. We denote by $\Gamma_0$ the outer boundary curve of $G$, and by $\Gamma_1,\dots,\Gamma_n$ the internal boundary curves. The boundary-value problem that defines the warping function $u(x,G)$ of $G$ is

$$\begin{cases} \triangle u=-2 & \text{in } G,\\ u=0 & \text{on } \Gamma_0,\\ u=c_i & \text{on } \Gamma_i,\ i=1,\dots,n, \end{cases} \tag{1}$$

where the constants $c_i$ are determined by the conditions

$$\oint_{\Gamma_i}\frac{\partial u}{\partial n}\,ds=-2a_i,\quad i=1,\dots,n,$$

$\partial/\partial n$ is the inward normal derivative, and $a_i$ is the area enclosed by $\Gamma_i$.

A classical result of the theory of partial differential equations [9] is that the boundary-value problem (1) has a positive and unique solution.

☆ This work was partially supported by the Russian Foundation for the Basic Research (projects No. 11-01-00762-a, and No. 12-01-97013-r_Volga region_a).

*E-mail address:* rsalakhud@gmail.com.

0022-247X/$ – see front matter © 2013 Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jmaa.2013.08.042